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Harmonic oscillator subject to parametric pulses: an amplitude (Milne) oscillator approach

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Abstract

A harmonic oscillator subject to a parametric pulse is examined. The aim of the paper is to present a new theory for analysing transitions caused by parametric pulses. The new theoretical notions which are introduced relate the pulse parameters in a direct way with transition matrix elements.

The harmonic-oscillator transitions are expressed in terms of the asymptotic properties of a companion oscillator, the Milne (amplitude) oscillator. A traditional phase-amplitude decomposition of the harmonic-oscillator solutions results in the so-called Milne's equation for the amplitude, and the phase is determined by an exact relation to the amplitude. This approach is extended in the present analysis with new relevant concepts and parameters for pulse dynamics of classical and quantal systems.

The amplitude oscillator has a particularly nice numerical behaviour. In the case of strong pulses it does not possess any of the fast oscillations induced by the pulse on the original harmonic oscillator. Furthermore, the new dynamical parameters introduced in this approach are closely related to the relevant characteristics of the pulse.

The relevance to quantum mechanical problems such as reflection and transmission from a localized well and the mechanical problem of controlling vibrations is illustrated.

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(Some figures in this article are in colour only in the electronic version; see www.iop.org)

1. Introduction

The theory of parametrically excited harmonic oscillators appears in various text in classical mechanics [1–6] and is also at the heart of quantum mechanics. The typically studied parametric systems in classical mechanics are subject to periodic excitations due to the frequent occurrence of rotating machines and interests in the stability of such systems. In quantum mechanics similar equations refer to electron dynamics in atom lattices or in constant amplitude periodic external fields.

In this paper we focus on *pulsed* parametric excitations of the harmonic oscillator. Similar situations can be identified for example in (finite-time) manoeuvred flexible multi-body systems and electronic states subject to pulsed-field excitations. In the context of 'smart materials', parametric control of vibrations is also of interest [6–9]. Often in classical mechanics external and parametric excitations go together. In this respect we neglect the direct response of the oscillator and concentrate our analysis on the homogeneous equation.

When designing particular pulses it is of interest to develop a theoretical framework which closes the gap between pulse parameters and transitions. This will be the main focus of this paper.

We explore the well known analysis of waves and oscillations in terms of amplitudes and phases [9–15], in which the nonlinear equation for the amplitude, the Milne equation, will play a key role. The theoretical result of our amplitude-phase analysis is an exact formalism with numerical and interpretational advantages compared to straightforward integration of the parametric oscillator equation. Not only are the computations faster, but the new dynamical parameters correspond closely to the relevant properties of the parametric pulse such as: duration, strength, and 'frequency-pulse area'.

The paper is organized as follows. Section 2 introduces the amplitude-phase decomposition of the parametric oscillator solutions. We explore the nonlinear Milne equation for the amplitude function and focus on its properties as $t \to \pm\infty$. In section 3 we write the fundamental solutions of the parametric oscillator in terms of the amplitude function and we investigate its properties as $t \to \pm\infty$. The important transition matrices for real cos/sin solutions and propagating complex solutions are derived and discussed in section 4. Section 5 is devoted to a numerical study of a $\cos^2$-pulse and conclusions are drawn in section 6.

2. The parametrically pulsed oscillator and the corresponding Milne oscillator

The parametric oscillator can be written as

$$\ddot{x} + \omega^2(t)x = 0 \tag{1}$$

where $\omega(t)(>0)$. In this paper we focus on the symmetric functions $\omega(t)$, which simplifies the derivations and further assume the normalized asymptotic limits

$$\omega(t) \to 1 \qquad \text{as} \quad t \to \pm\infty. \tag{2}$$

This unit limit of the symmetric angular frequency function $\omega(t)$ is obtained in general by a constant scaling of time. The non-constant behaviour of $\omega(t)$ is assumed to be sufficiently localized in time and will be referred to as the *pulse*.

The standard analysis of the pulse dynamics of the oscillator (1) would be to rewrite the equation as two first-order equations subject to relevant initial and final linear combinations of fundamental solutions of the asymptotically 'free' harmonic oscillator as $t \to \pm\infty$. Initially the parametric oscillator behaves like a harmonic oscillator of unit angular frequency. After the action of the pulse it again resumes the harmonic-oscillator behaviour with unit frequency, but the linear combinations of fundamental solutions have changed.

Instead of numerically integrating the original oscillator equations and extracting the transition properties directly as numbers we propose another approach. Some new theoretical notions are introduced which relate the pulse parameters in a direct way with the transition matrix elements. For this purpose an amplitude-phase ansatz for the oscillator solutions are introduced as

$$x_\pm = \rho \exp(\pm \mathrm{i}\theta) \tag{3}$$

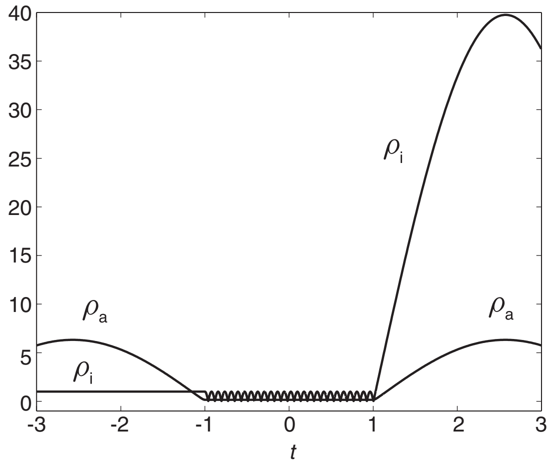

Figure 1. Two possible amplitude (Milne) solutions for the square-pulse model described in example 1 with parameters $\kappa = 40$, $T = 1$. The pre-pulse amplitude ρ_i starts with $\rho_i = 1$ at $t = -\infty$, oscillates in the pulse region and increases to large amplitude oscillations for $t > T$. The adiabatic amplitude ρ_a is constant during the pulse and oscillates between $\kappa^{\pm 1/2}$ for $|t| > T$.

which has to satisfy (1). Inserting (3) into (1) we obtain the amplitude oscillator equation

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3} \tag{4}$$

provided the phase satisfies

$$\dot{\theta} = \rho^{-2}. \tag{5}$$

Despite its nonlinearity, the differential equation (4) (the so-called Milne equation) can be solved numerically without any problem. The (real!) solution is typically oscillatory but *much* less so than the solutions of equation (1). In particular, the solution has no zeros. Depending on the initial conditions for ρ and $\dot{\rho}$, various Milne solutions can be generated. It can be shown that *any* solution of the nonlinear differential equation (4) together with (5) generates a pair of fundamental solutions (3) of (1). To analyse the parametric oscillator behaviour we typically have three natural choices for the amplitude function: ρ_i, ρ_f and ρ_a. The *pre-pulse amplitude* ρ_i is defined to be asymptotically constant as $t \to -\infty$. From (2) and (4) we obtain the constant limit $\rho_i \to 1$, as $t \to -\infty$. Eventually the pulse excites oscillations in the amplitude (see figure 1). The *post-pulse amplitude* ρ_f is defined to be asymptotically constant as $t \to +\infty$. From (2) and (4) we obtain the constant limit $\rho_f \to 1$, as $t \to +\infty$. In this case the amplitude has to be oscillatory initially. A third solution ρ_a can be constructed which is as slowly varying as possible in the pulse region. This solution will be called the *adiabatic amplitude* ρ_a. An effective and easy way to find initial conditions for such an *exact* solution is to look at the simplest approximate solution of Milne's equation. We simply try to find an almost 'constant' solution ρ_a and neglect all time derivatives of this solution. Milne's equation then reduces to

$$\rho_a^4(t) \approx \frac{1}{\omega^2(t)}. \tag{6}$$

This approximation, known as the WKB approximation in context with the quantum wavefunction [15, 16], is exact only if ω^2 is truly constant. The ambition in this paper is just to find a slowly varying amplitude ρ in the region where the pulse is the largest. However, it does not have to be a unique adiabatic solution, but rather one in a family (see a recent discussion in [17]). The adiabatic solution in this paper is defined by integrating Milne's equation from the initial conditions $\rho(0) = 1/\sqrt{\omega(0)}$ and $\dot{\rho}(0) = 0$ based on (6). The very nice properties of the adiabatic amplitude for strong pulses motivates the search for its relation to the original parametric oscillator.

The formal Hamiltonian governing the dynamics of the Milne solutions is

$$H(p_\rho, q, t) = \tfrac{1}{2}p_\rho^2 + \tfrac{1}{2}\omega^2(t)\rho^2 + \tfrac{1}{2}\rho^{-2} \tag{7}$$

where $p_\rho = \dot{\rho}$. The energy (7) is constant before and after the pulse and we introduce the asymptotic energy E_{M} for the asymptotic Milne solutions q

$$E_{\mathrm{M}} = \tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}q^2 + \tfrac{1}{2}q^{-2}. \tag{8}$$

The asymptotic amplitude oscillator is governed by the particular potential function $V(q) = \frac{1}{2}(q^2 + q^{-2})$ possessing a single minimum at $q = 1$. In this picture the amplitude energy is bounded by $E_{\mathrm{M}} \geqslant 1$, with the minimum energy attained by the stationary amplitudes $q = \rho_i$ and $q = \rho_f$, but only in the initial and final asymptotic regions, respectively.

We now seek the particular time dependence for non-stationary asymptotic amplitudes. Since energy is conserved in the asymptotic regions, it is in fact possible to solve the time dependence explicitly. To see this, we consider the single second-order differential equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + q = \frac{1}{q^3}. \tag{9}$$

It is instructive to see this geometrically as the radial equation for a two-dimensional radially symmetric linear oscillator, i.e. with $\boldsymbol{r} = (x, y)$

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} + \boldsymbol{r} = \boldsymbol{0}. \tag{10}$$

With this view, the amplitude (or radial variable) q is of the form

$$q = \sqrt{x^2(t) + y^2(t)}. \tag{11}$$

With a suitable shift of the origin of time t the orbit can be made to coincide with the large and small symmetry axes, rigidly rotated in the x-, y-plane. Then the form simplifies to

$$q = \sqrt{q_\pm^2\cos^2(t-\tau') + q_\mp^2\sin^2(t-\tau')} \tag{12}$$

where $q_\pm$ are the turning points of the amplitude oscillator. Solving the turning point equation ($\dot{q} = 0$ in equation (8)) for the asymptotic amplitude motion, we find

$$q_\pm = \sqrt{E_{\mathrm{M}} \pm \sqrt{E_{\mathrm{M}}^2 - 1}}. \tag{13}$$

A further simplification using trigonometric relations then yields

$$q(t) = \sqrt{E_{\mathrm{M}} \pm \sqrt{E_{\mathrm{M}}^2 - 1}\,\cos(2(t-\tau'))}. \tag{14}$$

A suitable origin $t = \tau'$ may be chosen conveniently at the larger or smaller turning point with + or −, respectively.

We can now describe the asymptotic behaviour of any amplitude on either side of the pulse region. It may not be possible to find the connections across the pulse region in a nice form, however. For example, given the pre-pulse amplitude ρ_i, we are not able to write down the parameters $E_{\mathrm{M}}^{(i)}$ and τ_i', as $t \to +\infty$. On the other hand, if we have found $E_{\mathrm{M}}^{(a)}$ and τ_a' for an adiabatic and symmetric amplitude ρ_a as $t \to +\infty$, then the symmetry requires $E_{\mathrm{M}}^{(a)}$ and $-\tau_a'$ as $t \to -\infty$.

Example 1. Let us illustrate the theory for the case of a square pulse

$$\omega = \kappa \ (>1) \qquad -T < t < T \qquad \text{and} \qquad \omega = 1 \quad |t| \geqslant T. \tag{15}$$

The adiabatic amplitude is $\rho_a = 1/\sqrt{\kappa}$, $-T < t < T$, and the asymptotic energy $E_{\mathrm{M}} = (1+\kappa^2)/(2\kappa)$. At $t = \pm T$ the amplitude smoothly assumes its asymptotic forms. It increases from a minimum (at an inner turning point in the potential well) so that $\tau' = T$. For $t > T$ we have

$$q_a(t) = \sqrt{E_{\mathrm{M}} - \sqrt{E_{\mathrm{M}}^2 - 1}\ \cos(2(t-T))}. \tag{16}$$

It is straightforward to verify $q_a(T) = 1/\sqrt{\kappa}$ and also $q_\pm = \kappa^{\pm 1/2}$.

We return in the next section to the harmonic oscillator and its asymptotic behaviour.

3. The asymptotic form of the harmonic-oscillator solutions

Any solution of the parametrically pulsed harmonic oscillator is a linear combination of cos t and sin t, as $t \to \pm\infty$. In the following we restrict the analysis to symmetric pulses and a symmetric amplitude function with the asymptotic form:

$$\rho_a \to q_a = \sqrt{E_{\mathrm{M}} - \sqrt{E_{\mathrm{M}}^2 - 1}\ \cos(2(t-\tau'))} \qquad t \to +\infty. \tag{17}$$

Our amplitude-phase decomposition using an adiabatic symmetric amplitude suggests real fundamental solutions of the form

$$C(t) = \rho_a(t)\cos\left(\int_0^t \rho_a^{-2}(t')\,\mathrm{d}t'\right) \qquad S(t) = \rho_a(t)\sin\left(\int_0^t \rho_a^{-2}(t')\,\mathrm{d}t'\right). \tag{18}$$

To analyse the corresponding asymptotic form as $t \to +\infty$, we introduce the phase difference, α, between exact and asymptotic amplitude forms

$$\alpha = \int_0^{+\infty} (\rho_a^{-2}(t') - q_a^{-2}(t'))\,\mathrm{d}t'. \tag{19}$$

We then get

$$C(t) \to q_a(t)\cos\left(\int_0^t q_a^{-2}(t')\,\mathrm{d}t' + \alpha\right) \qquad S(t) \to q_a(t)\sin\left(\int_0^t q_a^{-2}(t')\,\mathrm{d}t' + \alpha\right). \tag{20}$$

The evolving phase in these functions is clearly not linear. A symbolic evaluation[1] gives

$$\int_0^t q_a^{-2}(t')\,\mathrm{d}t' = \tan^{-1}\left(\left[E_{\mathrm{M}} + \sqrt{E_{\mathrm{M}}{}^2 - 1}\right]\tan(t-\tau')\right) + \tan^{-1}\left(\left[E_{\mathrm{M}} + \sqrt{E_{\mathrm{M}}{}^2 - 1}\right]\tan\tau'\right). \tag{21}$$

Collecting the additional constant phase appearing in (21) we thus have the asymptotic forms

$$C(t) \to q_a(t)\cos(\phi_a(t) + \Delta + \alpha) \qquad S(t) \to q_a(t)\sin(\phi_a(t) + \Delta + \alpha) \tag{22}$$

with

$$\phi_a(t) = \int_{\tau'}^t q_a^{-2}(t')\,\mathrm{d}t' = \tan^{-1}\left(\left[E_{\mathrm{M}} + \sqrt{E_{\mathrm{M}}{}^2 - 1}\right]\tan(t-\tau')\right) \tag{23}$$

$$\Delta = \int_0^{\tau'} q_a^{-2}(t')\,\mathrm{d}t' = \tan^{-1}\left(\left[E_{\mathrm{M}} + \sqrt{E_{\mathrm{M}}{}^2 - 1}\right]\tan(\tau')\right). \tag{24}$$

[1] MAPLE: q(t) := sqrt(Em − sqrt(Em*Em − 1)*cos(2*(t − delta))); simplify(combine(int(1/(q(t)*q(t)), t = 0..T), trig)).

For the moment we neglect the constant phases and try to simplify the reduced asymptotic forms

$$c(t) = q_a(t)\cos\phi_a(t) \qquad s(t) = q_a(t)\sin\phi_a(t). \tag{25}$$

A further symbolic analysis reveals

$$c(t) = \sqrt{E_{\mathrm{M}} - \sqrt{E_{\mathrm{M}}^2 - 1}}\,\cos(t - \tau') \tag{26}$$

and

$$s(t) = \sqrt{E_{\mathrm{M}} + \sqrt{E_{\mathrm{M}}^2 - 1}}\,\sin(t - \tau'). \tag{27}$$

The energy factors here are nothing more than the turning point positions. We recall their definitions

$$q_\pm = \sqrt{E_{\mathrm{M}} \pm \sqrt{E_{\mathrm{M}}^2 - 1}}. \tag{28}$$

At this point we can start tracing the relations back to the original real amplitude phase solutions. From (22) and (25)–(27) we have the matrix relation

$$\lim_{t\to+\infty}(C(t), S(t)) = \cos(t - \tau'), (\sin(t - \tau'))$$
$$\times \begin{pmatrix} q_- & 0 \\ 0 & q_+ \end{pmatrix} \begin{pmatrix} \cos(\Delta + \alpha) & \sin(\Delta + \alpha) \\ -\sin(\Delta + \alpha) & \cos(\Delta + \alpha) \end{pmatrix}. \tag{29}$$

This is referred to as the τ'-dressed representation in the limit $t \to +\infty$. In the opposite limit $t \to -\infty$, the symmetry requires

$$\lim_{t\to-\infty}(C(t), S(t)) = (\cos(t + \tau'), \sin(t + \tau'))$$
$$\times \begin{pmatrix} q_- & 0 \\ 0 & q_+ \end{pmatrix} \times \begin{pmatrix} \cos(\Delta + \alpha) & -\sin(\Delta + \alpha) \\ \sin(\Delta + \alpha) & \cos(\Delta + \alpha) \end{pmatrix}. \tag{30}$$

We notice here that the true dynamical phases are τ' and $\alpha + \Delta$.

Example 2. For the case of a square pulse defined in example 1, where $\tau' = T$, we can analytically evaluate the constant phases α and Δ

$$\alpha = \int_0^T \left(\kappa - \left(E_{\mathrm{M}} - \sqrt{E_{\mathrm{M}}^2 - 1}\,\cos(2(t - T))\right)^{-1}\right)\mathrm{d}t = \kappa T - \Delta \tag{31}$$

with

$$\Delta = \tan^{-1}\left(\left[E_{\mathrm{M}} + \sqrt{E_{\mathrm{M}}{}^2 - 1}\right]\tan(T)\right). \tag{32}$$

The turning point positions are

$$q_- = \sqrt{E_{\mathrm{M}} - \sqrt{E_{\mathrm{M}}^2 - 1}} = 1/\sqrt{\kappa} \qquad q_+ = \sqrt{E_{\mathrm{M}} + \sqrt{E_{\mathrm{M}}^2 - 1}} = \sqrt{\kappa}. \tag{33}$$

We note that $E_{\mathrm{M}} + \sqrt{E_{\mathrm{M}}^2 - 1}$ is the maximum value (κ) of the square pulse. Furthermore, $2\tau'$ is the range (or duration) and $2(\alpha + \Delta)$ is the area of the square pulse.

4. Pulse-induced transition matrices

The real transition matrix due to the pulse can now be constructed from the forward- and backward-connections in the previous section. To simplify the notation we introduce one symbol for the dynamical phase $\alpha + \Delta$

$$\delta' = \alpha + \Delta. \tag{34}$$

The τ'-dressed forward- and backward-connections are now defined as matrices

$$\boldsymbol{T}_+ = \begin{pmatrix} q_- \cos\delta' & q_- \sin\delta' \\ -q_+ \sin\delta' & q_+ \cos\delta' \end{pmatrix} \tag{35}$$

$$\boldsymbol{T}_- = \begin{pmatrix} q_- \cos\delta' & -q_- \sin\delta' \\ q_+ \sin\delta' & q_+ \cos\delta' \end{pmatrix}. \tag{36}$$

They combine to a τ'-dressed overall transition

$$\boldsymbol{M}_{\tau'} = \boldsymbol{T}_+ \boldsymbol{T}_-^{-1} = \begin{pmatrix} \cos 2\delta' & q_-^2 \sin 2\delta' \\ -q_+^2 \sin 2\delta' & \cos 2\delta' \end{pmatrix}. \tag{37}$$

In the evaluations here we have used the fact that $q_+ q_- = 1$. As a consequence we see that $\det \boldsymbol{M}_{\tau'} = 1$.

The pulse-induced transitions for *undressed* cos/sin oscillations are then given by the matrix elements of $\boldsymbol{M}$, where

$$\boldsymbol{M} = \boldsymbol{D}(\tau')\boldsymbol{M}_{\tau'}\boldsymbol{D}(\tau') \qquad \boldsymbol{D}(\tau') = \begin{pmatrix} \cos\tau' & -\sin\tau' \\ \sin\tau & \cos\tau' \end{pmatrix}. \tag{38}$$

In explicit terms we have

$$\boldsymbol{M} = \begin{pmatrix} E_{\mathrm{M}} \sin\tau \sin\delta + \cos\tau \cos\delta & E_{\mathrm{M}} \cos\tau \sin\delta - \sin\tau \cos\delta - \sqrt{E_{\mathrm{M}}^2 - 1}\, \sin\delta \\ -E_{\mathrm{M}} \cos\tau \sin\delta + \sin\tau \cos\delta - \sqrt{E_{\mathrm{M}}^2 - 1}\, \sin\delta & E_{\mathrm{M}} \sin\tau \sin\delta + \cos\tau \cos\delta \end{pmatrix} \tag{39}$$

where $\tau = 2\tau'$ and $\delta = 2\delta'$. This is a primary result for transitions of real cos/sin solutions.

We note that the matrix elements are restricted by two conditions $\det \boldsymbol{M} = 1$ and $M_{11} = M_{22}$. Just two real parameters are needed to describe the matrix. In our analysis we have three parameters $E_{\mathrm{M}}, \delta, \tau$. The main disadvantage with any choice of dynamical parameters is a possible lack of monotonicity with respect to pulse parameters. For the square pulse model (with its two parameters T and κ) we have found a monotonic relation to $E_{\mathrm{M}}, \delta, \tau$. An arbitrary parametrization such as

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ (a^2 - 1)/b & a \end{pmatrix} \tag{40}$$

would not provide us with the well behaved parameters a and b and in this respect three parameters are better than two. A more sophisticated parametrization is perhaps

$$\boldsymbol{M} = \begin{pmatrix} \cos y \cosh x & \sin y \cosh x - \sinh x \\ -\sin y \cosh x - \sinh x & \cos y \cosh x \end{pmatrix}. \tag{41}$$

Having the square-pulse $\boldsymbol{M}$ matrix as an exact reference we know that the elements contain two real phases and an amplitude parameter. It is not obvious in the sophisticated parametrization to see that they are the real and non-oscillating parameters x and y.

Example 3. In analogy with previous examples we can substitute $\delta = 2\kappa T$, $\tau = 2T$, and $E_{\mathrm{M}} = (1 + \kappa^2)/(2\kappa)$ to find the exact 'pulse excitation' matrix $\boldsymbol{M}$ for the *square pulse*. In the matrix elements, only δ depends on the strength and the duration of the pulse. In fact, it *is*

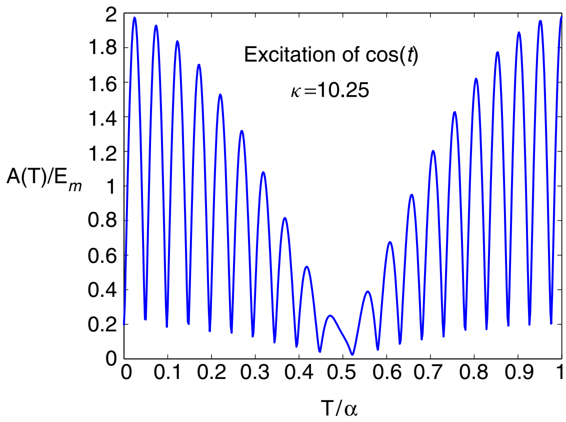

Figure 2. A $\cos t$ oscillation is excited by a square pulse of fixed strength $\kappa = 10.25$ but with a variable duration. The amplitude of the excited oscillation contains slow and fast variations with 'global' and 'local' maxima, respectively. The amplitude has been reduced by a factor $E_{\rm M}$ for comparison with our estimate of global $A_{\rm max}$.

rather the angular frequency-pulse action (an action integral). Furthermore, $E_{\rm M}$ only depends on the strength of the pulse, and τ is the pulse duration. If we study large, massive pulses it is relevant to see δ as the most sensitive dynamical quantity.

Let us analyse the amplitude amplification of a pure cosine oscillation by a strong pulse. In this context we assume $E_{\rm M} \gg 1$ and simplify the matrix elements:

$$\boldsymbol{M} \approx \begin{pmatrix} E_{\rm M} \sin\tau \sin\delta & E_{\rm M} \cos\tau \sin\delta - E_{\rm M} \sin\delta \\ -E_{\rm M} \cos\tau \sin\delta - E_{\rm M} \sin\delta & E_{\rm M} \sin\tau \sin\delta \end{pmatrix}. \tag{42}$$

The first column of the $\boldsymbol{M}$-matrix gives us the cos/sin sub-amplitudes of the excited oscillation. We then have the total amplitude of the excited oscillation

$$A = \sqrt{2} E_{\rm M} \, |\sin\delta| \, \sqrt{1 + \cos\tau}. \tag{43}$$

How large can this amplitude be? Obviously

$$A_{\rm max} \approx 2E_{\rm M} \tag{44}$$

with the ideal conditions for δ and τ being

$$|\sin\delta| \approx 1 \qquad \cos\tau \approx 1. \tag{45}$$

Let us then design a square pulse which gives this large amplitude. The pulse duration can be taken as $\tau = 2\pi$, i.e. $T = \pi$ in our model. Then we solve for a 'large' κ in the last equation, which gives

$$\kappa = (2n+1)/4 \qquad n \gg 1. \tag{46}$$

In figure 2 we show how the exact amplitude (obtained without approximation (42)) of the excited oscillation depends on the pulse duration $\tau = 2T$ with a fixed value of κ (and $E_{\rm M}$). We have chosen a value of κ from the estimate above with $n = 20$ and can confirm that a 'global' (and 'local') maximum is located close to $T = \pi$. We also see that $A_{\rm max}$ is close to our estimate (44). In figure 3 we compare the action of the constructed pulse on a $\cos t$ solution with that of a pulse which has approximately half the duration.

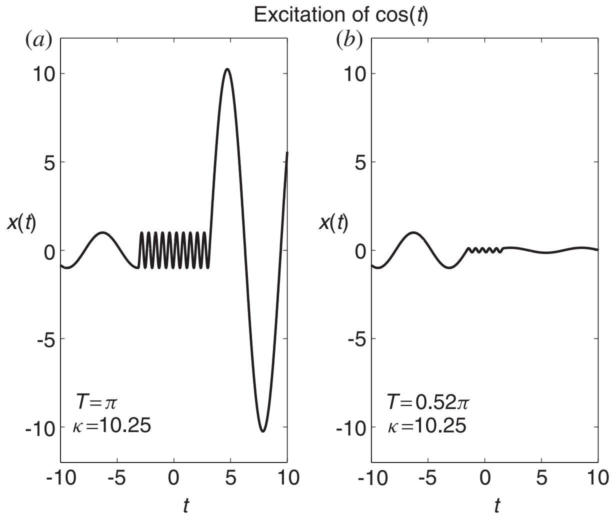

Figure 3. A $\cos t$ oscillation is excited by two parametric square pulses of the same strength $\kappa = 10.25$. (*a*) Amplification for $T = \pi$ where the pulse duration is 2π. (*b*) Quenching for $T = 0.52\pi$.

We conclude this section by discussing the pulse-induced transition matrix for propagating (complex) oscillations $\exp(\pm \mathrm{i}t)$. The propagating asymptotic oscillations are linear combinations of our real cos/sin solutions, i.e.

$$(\exp(\mathrm{i}t), \exp(-\mathrm{i}t)) = (\cos t, \sin t) \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} = (\cos t, \sin t)\, \boldsymbol{C}. \tag{47}$$

Therefore the corresponding complex transition matrix for these solutions is expressed as

$$\boldsymbol{P} = \boldsymbol{C}^{-1}\boldsymbol{M}\boldsymbol{C}. \tag{48}$$

Recalling equation (38) we may also write the transition matrix as

$$\boldsymbol{P} = \left[\boldsymbol{D}(-\tau')\boldsymbol{C}\right]^{-1} \boldsymbol{M}_{\tau'}\boldsymbol{D}(\tau')\boldsymbol{C} \tag{49}$$

or

$$\boldsymbol{P} = \begin{pmatrix} \exp(-\mathrm{i}\tau') & 0 \\ 0 & \exp(\mathrm{i}\tau') \end{pmatrix} \boldsymbol{C}^{-1}\boldsymbol{M}_{\tau'}\boldsymbol{C} \begin{pmatrix} \exp(-\mathrm{i}\tau') & 0 \\ 0 & \exp(\mathrm{i}\tau') \end{pmatrix}. \tag{50}$$

The middle three matrices combine to

$$\boldsymbol{P}_{\tau'} = \begin{pmatrix} \cos 2\delta' + \mathrm{i}E_{\mathrm{M}}\,\sin 2\delta' & \mathrm{i}\sqrt{E_{\mathrm{M}}^2 - 1}\,\sin 2\delta' \\ -\mathrm{i}\sqrt{E_{\mathrm{M}}^2 - 1}\,\sin 2\delta' & \cos 2\delta' - \mathrm{i}E_{\mathrm{M}}\,\sin 2\delta' \end{pmatrix} \tag{51}$$

where

$$(q_\pm^2 + q_\pm^{-2})/2 = E_{\mathrm{M}} \qquad (q_\pm^2 - q_\pm^{-2})/2 = \pm\sqrt{E_{\mathrm{M}}^2 - 1}. \tag{52}$$

Including the framing matrices and adopting the simplifications $\delta = 2\delta',\ \tau = 2\tau'$ we find

$$\boldsymbol{P} = \begin{pmatrix} (\cos\delta + \mathrm{i}E_{\mathrm{M}}\,\sin\delta)\mathrm{e}^{-\mathrm{i}\tau} & \mathrm{i}\sqrt{E_{\mathrm{M}}^2 - 1}\,\sin\delta \\ -\mathrm{i}\sqrt{E_{\mathrm{M}}^2 - 1}\,\sin\delta & (\cos\delta - \mathrm{i}E_{\mathrm{M}}\,\sin\delta)\mathrm{e}^{\mathrm{i}\tau} \end{pmatrix}. \tag{53}$$

The complex matrix $\boldsymbol{P}$ is particularly suited for quantum mechanical analyses.

Example 4. In analogy with the previous examples we can, with the identification $t = x$, analyse transmission and reflection of quantal waves from a square potential well. Using the complex transition matrix in (53) we find the boundary conditions

$$\exp(-\mathrm{i}x) \qquad x \to -\infty \tag{54}$$

$$(\cos\delta - \mathrm{i}E_\mathrm{M}\sin\delta)\mathrm{e}^{\mathrm{i}\tau}\exp(-\mathrm{i}x) + \mathrm{i}\sqrt{E_\mathrm{M}^2 - 1}\sin\delta\exp(\mathrm{i}x) \qquad x \to +\infty. \tag{55}$$

Normalizing the left propagating current in the limit as $x \to +\infty$, we find the transmission and reflection amplitudes

$$R = \frac{\mathrm{i}\sqrt{E_\mathrm{M}^2 - 1}\sin\delta\,\mathrm{e}^{-\mathrm{i}\tau}}{\cos\delta - \mathrm{i}E_\mathrm{M}\sin\delta} \tag{56}$$

$$T = \frac{\mathrm{e}^{-\mathrm{i}\tau}}{\cos\delta - \mathrm{i}E_\mathrm{M}\sin\delta}. \tag{57}$$

In the case of a quantum mechanical square well problem all we have to do are the substitutions derived in the previous examples. The result is exact.

5. A smooth pulse

We demonstrate some numerical results here for a smooth *parameter*-pulse of finite range

$$\omega^2(t) = 1 + (\beta^2 - 1)\cos^2\left(\frac{\pi t}{2T}\right) \qquad -T \leqslant t \leqslant T. \tag{58}$$

The maximal strength of the *frequency* pulse is $\beta > 1$, which can be compared with the square-pulse magnitude $\kappa > 1$.

We use half the pulse range (i.e. $0 \leqslant t \leqslant T$) in the calculations of the adiabatic Milne solution $\rho_a(t)$. The Milne energy E_M is calculated from $\rho_a(T)$ and $\dot\rho_a(T)$ according to (8). The dynamical-range parameter τ is obtained from the relation

$$\dot q_a(T) q_a(T) = \sqrt{E_\mathrm{M}^2 - 1}\sin(2T - \tau) \tag{59}$$

which is easy to verify from the chosen asymptotic form

$$q_a(t) = \sqrt{E_\mathrm{M} - \sqrt{E_\mathrm{M}^2 - 1}\cos(2t - \tau)}. \tag{60}$$

The dynamical phase shift δ requires a numerical integration of $\rho_a^{-2}(t)$, which can be easily carried out together with the computation of $\rho_a(t)$ and $\dot\rho_a(t)$, and is then obtained from

$$\delta = 2\int_0^T \rho_a^{-2}(t)\,\mathrm{d}t - 2\tan^{-1}\left(\left[E_\mathrm{M} + \sqrt{E_\mathrm{M}{}^2 - 1}\right]\tan(T - \tau/2)\right). \tag{61}$$

In a first application, we calculate our three dynamical parameters δ, τ, and E_M as a function of the pulse amplitude β in the interval $0 \leqslant \beta \leqslant 10$. We take $T = 1$ and obtain the results shown in the two upper sub-plots of figure 4.

They show a smooth behaviour of δ, τ, and E_M, where δ increases almost linearly with β as for the square pulse. The slope is seen to be considerably smaller. τ is almost independent of β, which also agrees with the square-pulse case. The Milne energy E_M is slightly less linear. We note that the values of E_M are still moderately large for $\beta = 10$.

The last sub-plot of figure 4 shows the β dependence of the matrix element M_{11}. The eigenvalues of the $\boldsymbol{M}$ matrix can be written explicitly in terms of this matrix element

$$m_\pm = M_{11} \pm \sqrt{M_{11}^2 - 1} \qquad m_+ m_- = 1 \tag{62}$$

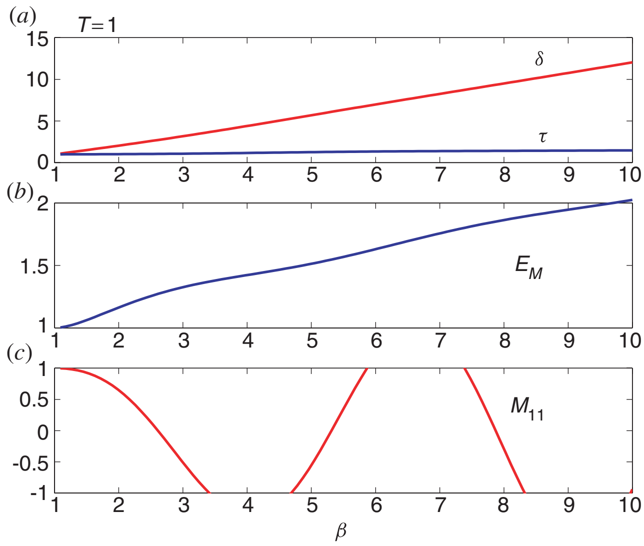

Figure 4. (*a*), (*b*) Numerically calculated behaviour of δ, τ, and E_{M} as functions of β. (*c*) The part of the oscillatory matrix element M_{11} corresponding to regular (non-magnifying) transitions. $T = 1$.

where

$$M_{11} = E_{\mathrm{M}} \sin\tau \sin\delta + \cos\tau \cos\delta. \tag{63}$$

M_{11} can be either positive or negative depending on the phases δ and τ. For the case

$$|M_{11}| \leqslant 1 \tag{64}$$

the eigenvalues (and eigensolutions) are complex conjugates with $|m_\pm| = 1$. This is the typical, regular behaviour if $E_{\mathrm{M}}(>1)$ is close to 1, i.e. small values of β in figure 4. As β (and E_{M}) grows, the regular behaviour will disappear and reoccur depending on the oscillatory behaviour of M_{11}. At the extreme values $M_{11} = \pm 1$ the eigenvalues become truly real in intervals that grow as β becomes larger. The real eigenvalues and the possible amplification effect is related to the largeness of E_{M}. The condition for such intervals is

$$|M_{11}| > 1. \tag{65}$$

The larger the values of E_{M}, the wider the range of β where oscillations are possibly amplified. A very specific oscillation of an eigensolution can in fact be reduced in the same interval of β while all the others are amplified.

To continue our investigation, we fix the value of the pulse strength to $\beta = 10$ and vary the pulse duration ($0 < 2T < 10$). The result is shown in figure 5. Here the phases δ and τ show an almost linear dependence. The Milne energy E_{M}, however, seems to have a clear peak near the limit of short, 'sudden' pulses. We recall here that the square pulse had a value of $E_{\mathrm{M}} = (\kappa^2+1)/(2\kappa) \approx 5.05$, which is independent of the pulse duration. This value is close to the limiting peak value in figure 5 of the $\cos^2$-pulse.

The matrix element M_{11} in the last sub-plot of figure 5 has more oscillations than in the previous plot. Although the Milne energy E_{M} is largest for short pulses, the corresponding behaviour of M_{11} is regular and the eigenvalues are of unit magnitude.

Let us finally discuss how to construct a parametric $\cos^2$-pulse that minimizes an existing oscillation in the harmonic oscillator. For this purpose we first try to derive the ideal conditions in terms of our dynamical parameters E_{M}, δ and τ.

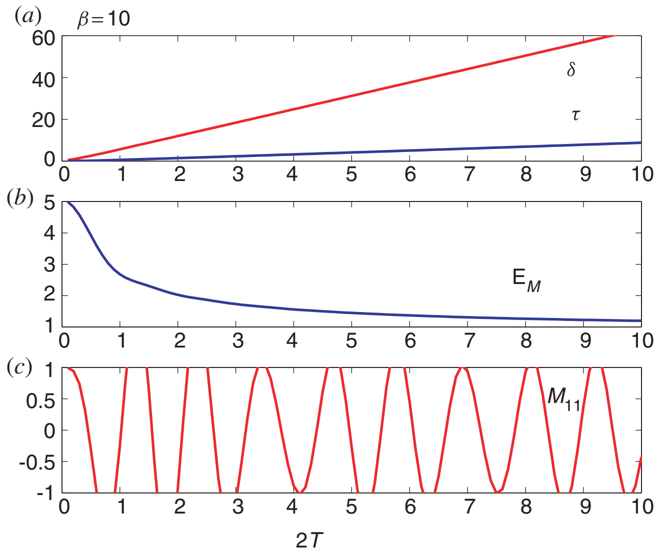

Figure 5. (a), (b) Numerically calculated behaviour of δ, τ, and E_{M} as functions of $2T$. (c) The part of the oscillatory matrix element M_{11} corresponding to regular (non-magnifying) transitions. $\beta = 10$.

We require a sufficiently large magnitude of the element M_{11}. It is possible by simple means to verify that $-E_{\mathrm{M}} \leqslant M_{11} \leqslant E_{\mathrm{M}}$. The maximum magnitude is attained if

$$\sin\tau = \pm 1 \qquad \sin\delta = \pm 1 \qquad \text{(any combination).} \tag{66}$$

The eigenvalues and the possible magnifications are found to be

$$(m_+,\ m_-) = \left(\pm\left(E_{\mathrm{M}} + \sqrt{E_{\mathrm{M}}^2 - 1}\right), \pm\left(E_{\mathrm{M}} - \sqrt{E_{\mathrm{M}}^2 - 1}\right)\right) \tag{67}$$

where the signs go together. The maximal reduction factor $|m_-|$ is obtained for the eigensolution pertaining to m_-.

To see which oscillation is the corresponding eigenoscillation for the minimum eigenvalue we take a second look at the $\boldsymbol{M}$-matrix. The condition (66) can be optimized if we know that the shortest possible pulses give the largest Milne energy E_{M}. We may then restrict the condition to the smallest possible value of τ (the effective pulse-range parameter). Thus our explicit conditions take the form

$$\tau = \pi/2 \qquad \delta = (2n+1)\frac{\pi}{2} \qquad n = 0, 1, \ldots. \tag{68}$$

Inserting (68) into (39), we obtain

$$\boldsymbol{M} = \begin{pmatrix} \pm E_{\mathrm{M}} & \mp\sqrt{E_{\mathrm{M}}^2 - 1} \\ \mp\sqrt{E_{\mathrm{M}}^2 - 1} & \pm E_{\mathrm{M}} \end{pmatrix} \tag{69}$$

where the signs go together. We realize that the minimum eigenvalue oscillation needs to be proportional to $\sin(t + \pi/4) = (\cos t + \sin t)/\sqrt{2}$. Any oscillation of our harmonic oscillator can be seen as such an eigenoscillation by shifting the time origin, i.e. by a proper timing of the activation of the pulse. If this can be successfully achieved, we can go on and determine T and β of the pulse.

Table 1. Numerically calculated sequence of pulse parameters T and β satisfying the reduction condition (68). The Milne energy E_{M} and the eigenvalue m_- for the pulse-induced oscillator transitions are also given.

n	T	β	E_{M}	m_-
0	0.5233	0.4224	1.3847	0.427
1	1.3211	3.2073	1.2243	−0.518
2	1.1384	5.8632	1.5583	0.363
3	1.0792	8.5044	1.8413	−0.295
4	1.0329	11.2760	2.1135	0.252
5	1.0068	14.0348	2.3546	−0.223
6	0.9844	16.8630	2.5850	0.201
7	0.9692	19.6827	2.7961	−0.185
8	0.9555	22.5475	2.9990	0.172
9	0.9454	25.4061	3.1886	−0.161
10	0.9360	28.2970	3.3718	0.152
11	0.9286	31.1833	3.5452	−0.144
12	0.9218	34.0941	3.7134	0.137
13	0.9161	37.0012	3.8740	−0.131
14	0.9108	39.9278	4.0304	0.126
15	0.9063	42.8513	4.1808	−0.121
16	0.9020	45.7909	4.3275	0.117

We consider the sequence of solutions of (68) using a two-dimensional Newton root searching procedure. Hence, we determine numerically the corresponding pulse parameters which minimize the oscillation. The result is displayed in table 1. The reduction factor $|m_-|$ depends on E_{M} according to equation (67). The Milne energy increases in the sequence and the reduction factor decreases. The weakest pulse we found in the sequence does not fulfil our basic requirement $\beta > 1$ and is neglected in the discussion. For $n = 16$ we have $|m_-| \approx 0.12$ (cf figure 6). We note that the corresponding pulse strength parameter β is quite large and recall that the amplitude of the $\cos^2$-pulse depends quadratically on β. A square-shaped pulse of the same strength would be much more effective as a reduction tool, since E_{M} does not decrease with the pulse duration (cf figure 5) but stays at the peak level for all pulse durations. For a square pulse with the same strength as the $\cos^2$-pulse corresponding to $n = 16$ we would have $|m_-| \approx 0.02$.

6. Conclusions

The parametrically pulsed harmonic oscillator undergoes transitions of oscillation modes, which are interpreted and calculated from a particular, symmetric and well behaved adiabatic solution of an associated amplitude (Milne) oscillator. Calculations as well as interpretations of parametric excitations are intuitively powerful in this theoretical model. Three dynamical quantities E_{M} (where $E_{\mathrm{M}} + \sqrt{E_{\mathrm{M}}^2 - 1}$ is the effective pulse strength), δ (effective phase shift), and τ (effective pulse duration) are defined in terms of a symmetric and positive solution of the Milne oscillator. They parametrize the transition matrices while being closely related to the pulse properties.

A square-pulse model was studied analytically to illustrate the theoretical dynamical parameters in relation to the pulse parameters of interest here. We also studied a finite-range $\cos^2$-pulse by numerical computations to verify the relevance of the new dynamical parametrization as a tool in analysing pulse-induced transitions.

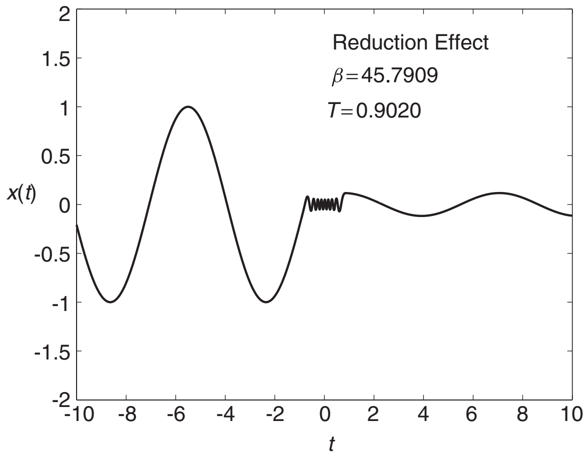

Figure 6. Illustration of a parametrically reduced oscillation.

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